A special case of \( p \)-groups

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Abstract

A group \( G \) is called a \( C_p \)-group, if \( G \) is a \( p \)-group for some prime \( p \), and

\[
Z(G) \leq G' < V(G) < G,
\]

where \( Z(G) \) is the center of \( G \) and \( G' \) is the commutator subgroup. It is a well known fact that a \( p \)-groups has a non trivial center. In this paper, we give some upper bound for the center of a \( C_p \)-group. We showed that if \( G \) is a \( C_p \)-group, then

\[
|Z(G)| \leq |G : Z(G)|.
\]

Moreover, if \( G \) has nilpotence class 3, then \( |Z(G)| \leq |G : G'| \).

Next, Let \( G \) be a finite group. We write \( \text{Irr}(G) \) for the set of irreducible characters of \( G \) and \( \text{nl}(G) = \{ \chi \in \text{Irr}(G) \mid \chi(1) \neq 1 \} \). Define the vanishing off subgroup of \( G \), by

\[
V(G) = \{ g \in G \mid \text{there exists } \chi \in \text{nl}(G) \text{ such that } \chi(g) \neq 0 \}.
\]

This subgroup was first introduced by Lewis in [1]. Note that \( V(G) \) is the smallest subgroup of \( G \) such that all nonlinear irreducible characters vanish on \( G \setminus V(G) \). Moreover, \( V(G) \) is a proper subgroup only if \( G \) is solvable (and of course nonabelian). A central series associated with the vanishing off subgroup, defined inductively by \( V_1 = V(G) \), and \( V_i = [V_{i-1}, G] \) for \( i \geq 2 \). Lewis proved in [1] that \( G_{i+1} \leq V_i \leq G_i \), and if \( V_i < G_i \), then \( V_j < G_j \) for all \( j \) such that \( 1 \leq j \leq i \). He also proved that if \( V_2 < G_2 \), then there exists a prime \( p \) such that \( G_i/V_i \) is an elementary abelian \( p \)-group for all \( i \geq 1 \). Also, consider the term \( G_i \) as the \( i \)-th term in the lower central series, which is defined by \( G_1 = G, G_2 = G' = [G,G], \) and \( G_i = [G_{i-1}, G] \) for \( i \geq 3 \). Consider \( Z_i \) as the \( i \)-th term in the lower central series, which is defined by \( Z_1 = Z(G) \), and \( Z_i/Z_{i-1} = Z(G/Z_{i-1}) \). For more details about these series see [2, 3].

We showed that if \( G \) be \( C_p \)-group and assume that \( G/Z(G) \) has exponent \( p^n \) for an integer \( n \geq 1 \) and \( Z(G) \) has exponent \( p \). If there exists \( x \in G \setminus V(G) \) such that \( x^{p^i} \not \in V(G) \) for \( i = 0, \ldots, n - 1 \) but \( x^{p^n} \in Z(G) \), then \( |Z(G)| < \sqrt{|G : Z(G)|} \).

Also, we proved that if \( G \) is a \( C_p \)-group, and \( Z(G) < (G'/Z_2) \), then there exists \( a \in (G'/Z_2) \setminus Z(G) \) such that \( a \) is conjugate to all of \( Z(G)a \). Hence, \( |Z(G)| \leq |G : G'Z_2| \).

References
