Exponents, exponents, exponents

Submission deadline: March 31^{st} 2018

Is the number

$$1 + 2^{2019^{2020}} + 3^{2019^{2020}} + \cdots + 2018^{2019^{2020}}$$

divisible by 2019?

The problem was solved by

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Discussion

Let $n = 2019^{2020}$ and

$$S = 1 + 2^n + 3^n + \cdots + 2017^n + 2018^n$$

By rearranging the terms, it can be seen that

$$S = (1 + 2018^n) + (2^n + 2017^n) + \cdots + (1009^n + 1010^n)$$

It is easy to see that the general term takes the form $p^n + (2019 - p)^n$ where $p = 1, 2, \cdots, 1009$.

Since 2019 is an odd integer, n is also an odd integer. Thus,

$$p^n + (2019 - p)^n = (p + (2019 - p))(p^{n-1} - p^{n-2}(2019 - p) + \cdots + (2019 - p)^{n-1}).$$

Therefore $p^n + (2019 - p)^n$ is divisible by 2019 for all p. Since S is the sum of such terms it easily follows that S is divisible by 2019.

Note The solution above uses the factorization

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$$

where n is odd. Instead, using the binomial theorem

$$(2019 - p)^n = 2019^n - C_1^n 2019^{n-1} p + C_2^n 2019^{n-2} p^2 + \cdots + C_{n-1}^n 2019 p^{n-1} + (-p)^n.$$

Since n is odd

$$p^n + (2019 - p)^n = 2019^n - C_1^n 2019^{n-1} p + C_2^n 2019^{n-2} p^2 + \cdots + C_{n-1}^n 2019 p^{n-1}.$$

Thus, $p^n + (2019 - p)^n$ is divisible by 2019.