A Pythagorean-like triple

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Find all possible integer solutions of

$$x^2 + y^2 + z^2 = 2xyz.$$

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Discussion

Clearly $x = 0, y = 0, z = 0$ is a solution. We will next show that this is the only integer solution.

Assume that x, y, z is a positive integer solution. Let 2^{q_1} be the highest power of 2 that divides x, 2^{q_2} be the highest power of 2 that divides y and 2^{q_3} be the highest power of 2 that divides z. Rename q_1, q_2, q_3 as p_1, p_2, p_3 so that $p_3 \leq p_2 \leq p_1$. Now x, y, z can written as $2^{p_1}a, 2^{p_2}b$ and $2^{p_3}c$ where a, b, c are odd integers.

Then we have

$$2^{2p_1}a^2 + 2^{2p_2}b^2 + 2^{2p_3}c^2 = 2^{p_1+p_2+p_3+1}abc \tag{1}$$

Then

$$2^{2(p_1-p_3)}a^2 + 2^{2(p_2-p_3)}b^2 + c^2 = 2^{p_1+p_2-p_3+1}abc \tag{2}$$

Now consider the following cases;

Case 1; $p_3 = p_1 = p_2$

In this case the left side of equation above is odd and right side of the equation is even.

Case 2; $p_3 < p_2$

In this case, the c^2 term is odd and all the others are even.

Case 3; $p_3 = p_2, p_3 < p_1$.

Let $b = 2k + 1$ and $c = 2s + 1$ then

$$2^{2(p_1-p_3)}a^2 + (2k+1)^2 + (2s+1)^2 = 2^{p_1+1}abc$$

Expanding the square terms we get an equation where all but one term are divisible by 4.

All cases above are clearly impossible and it is easy to see that the remaining two cases are also impossible. Thus $x = 0, y = 0, z = 0$ is the only solution.