A Pythagorean-like triple

Submission deadline: February 28th 2019

Find all possible integer solutions of

\[ x^2 + y^2 + z^2 = 2xyz. \]

The problem was solved by

- Ruben Victor Cohen Argentina.
- Abdulla Maseeh Alumni, American University of Sharjah, UAE.
- Hichem Zakaria Aichour
Discussion

Clearly $x = 0, y = 0, z = 0$ is a solution. We will next show that this is the only integer solution.

Assume that $x, y, z$ is a positive integer solution. Let $2^{q_1}$ be the highest power of 2 that divides $x$, $2^{q_2}$ be the highest power of 2 that divides $y$ and $2^{q_3}$ be the highest power of 2 that divides $z$. Rename $q_1, q_2, q_3$ as $p_1, p_2, p_3$ so that $p_3 \leq p_2 \leq p_1$. Now $x, y, z$ can written as $2^{p_1}a, 2^{p_2}b$ and $2^{p_3}c$ where $a, b, c$ are odd integers.

Then we have

$$2^{2p_1}a^2 + 2^{2p_2}b^2 + 2^{2p_3}c^2 = 2^{p_1 + p_2 + p_3 + 1}abc$$  \hspace{1cm} (1)

Then

$$2^{2(p_1 - p_3)}a^2 + 2^{2(p_2 - p_3)}b^2 + c^2 = 2^{p_1 + p_2 - p_3 + 1}abc$$  \hspace{1cm} (2)

Now consider the following cases;

Case 1: $p_3 = p_1 = p_2$
In this case the left side of equation above is odd and right side of the equation is even.

Case 2: $p_3 < p_2$
In this case, the $c^2$ term is odd and all the others are even.

Case 3: $p_1 = p_2, p_3 < p_1$.
Let $b = 2k + 1$ and $c = 2s + 1$ then

$$2^{2(p_1 - p_3)}a^2 + (2k + 1)^2 + (2s + 1)^2 = 2^{p_1 + 1}abc$$

Expanding the square terms we get an equation where all but one term are divisible by 4.

All cases above are clearly impossible and it is easy to see that the remaining two cases are also impossible. Thus $x = 0, y = 0, z = 0$ is the only solution.