An Odd Triangle

Submission deadline: May 31st 2019

Let T be an acute triangle. The area of T is denoted by A. If the length of each side of T is an odd prime number, show that $A^2 - \dfrac{3}{16}$ is an integer.

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Discussion

Let a, b and c be the lengths of sides. Then, using Heron's formula

$$4A = \sqrt{4(a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2)^2}$$

Let $a = 2n + 1, b = 2m + 1$ and $c = 2l + 1$. Now it is easy to see that a^2b^2 can be written in the form $4P + 1$ for some integer P. Similarly, $a^2c^2 = 4Q + 1$ and $b^2c^2 = 4R + 1$ for some integers Q, R. Therefore,

$$16A^2 = 4(4(P + Q + R) + 3) - (4(n(n + 1) + m(m + 1) + l(l + 1)) + 3)^2$$

where P, Q, R are integers. Since $n(n+1)+m(m+1)+l(l+1)$ is an even integer we have

$$16A^2 = 4(4(P + Q + R) + 3) - (8M + 3)^2$$

for some integer M. Thus

$$A^2 = P + Q + R - 4M^2 - 3M + \frac{3}{16}$$

Note: The result is true when the sides are odd integers and they do not have to be primes.