The Sum of Reciprocals

Submission deadline: December 31st 2019

Let $\{a_n\}$ be a sequence of numbers that satisfy $a_0 = 3$, and $(3 - a_{n+1})(6 + a_n) = 18$. Find $\sum_{i=0}^{n} \frac{1}{a_i}$.

The problem was solved by

- Jonatan Eklöv, *Berzeliusskolan, Sweden.*
- Ruben Victor Cohen, *Argentina.*
- Hari Kishan, *Department of Mathematics, D.N. College, Meerut, India.*
- Ievgen Murzak, *Ukraine.*
- Mohammed Fawzi Kharroub, *American University of Sharjah, UAE.*
- Samuel Mathew Tharakan.

Discussion: It is easy to see that

$$\frac{2}{a_n} + \frac{1}{3} = \frac{1}{a_{n+1}}.$$

Adding 1/3 to both sides of the equation above gives us

$$2\left(\frac{1}{a_n} + \frac{1}{3}\right) = \frac{1}{a_{n+1}} + \frac{1}{3}.$$

Thus, if $b_n = 1/a_n + 1/3$, then $b_{n+1} = 2b_n$. Hence,

$$\sum_{i=0}^{n} b_i = \frac{2}{3}(2^{n+1} - 1)$$

Therefore

$$\sum_{i=0}^{n} \frac{1}{a_i} = \frac{2}{3}(2^{n+1} - 1) - \frac{(n+1)}{3}$$