360

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Find all natural numbers n for which

$$n^2(n^2-1)(n^2-4)$$

is divisible by 360.

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Discussion:

Let $s = n^2(n^2-1)(n^2-4)$. For $n = 1$ or $n = 2$, it is easy to see that 360 divides s. Assume that $n > 2$.

Clearly,

$$s = (n-2) \cdot (n-1) \cdot n \cdot n \cdot (n+1) \cdot (n+2)$$

The prime factorization of 360 is $3^2 \cdot 2^3 \cdot 5$.

A group of 5 consecutive natural numbers contains a multiple of 5, therefore 5 divides s.

A group of 3 consecutive natural numbers contains a multiple of 3, and s is the product of two such groups. Thus, 3^2 divides s.

If n is even then 2^3 divides $n \cdot n \cdot (n+2)$.

If $n = 2k+1$, then $(n-1)(n+1) = 2^2 \cdot k \cdot (k+1)$. Hence 2^3 divides $(n-1)(n+1)$.

Thus $5 \cdot 3^2 \cdot 2^3$ divides s for all natural numbers.

Many solutions we received were variations of the solution above. However, Mr. Pankaj Chandra solved the problem by expressing the given term using ${}^{n+3}C_6$ and ${}^{n+2}C_5$.