Three Equations

Submission deadline: April 29^{th} 2020

Solve the following system of equations for x, y and z

$$x^2y^2 + x^2z^2 = axyz$$
$$y^2z^2 + y^2x^2 = bxyz$$
$$z^2x^2 + z^2y^2 = cxyz$$

where, a, b and c are given constants.

The problem was solved by

- Shubhan Bhatia, *Grade 12, GEMS Modern Academy, Dubai, UAE.*

- Sidharth Hariharan, *IB1, GEMS Modern Academy, Dubai, UAE.*

- Vansh Agarwal, *IB1, GEMS Modern Academy, Dubai, UAE.*

- Emre Karabıyık, *Hacettepe University, Faculty of Medicine, Ankara, Turkey.*

- Hari Kishan, *Department of Mathematics, D.N. College, Meerut, India.*

- Anya Bindra.

Discussion:

$$x^2y^2 + x^2z^2 = axyz \tag{1}$$
$$y^2z^2 + y^2x^2 = bxyz \tag{2}$$
$$z^2x^2 + z^2y^2 = cxyz \tag{3}$$

It is easy to see that if two of the variables are equal to 0, and the remaining one takes any value, then x, y, z is a solution. Moreover, it is not possible to have a solution where one variable is zero and the other two are non-zero. Therefore we will find solutions assuming each variable is non-zero.

Since $x \neq 0$, from (1) we get that

$$x = a\frac{yz}{y^2 + z^2} \tag{4}$$

Substitute (4) in (2). Then, since $yz \neq 0$, we get

$$a^2z^2 + (y^2 + z^2)^2 - ca(y^2 + z^2) = 0 \tag{5}$$

Next Substitute (4) in (3). Then, we get

$$y^2 + z^2 = \frac{1}{2}a(b + c - a) \tag{6}$$

Similarly we can get

$$z^2 + x^2 = \frac{1}{2}b(a + c - b) \tag{7}$$

$$x^2 + y^2 = \frac{1}{2}c(a + b - c) \tag{8}$$

From (6) and (7) we get that $x^2 - y^2 = (a - b)(a + b - c)/2$. Thus,

$$x^2 = \frac{1}{4}(a + b - c)(a + c - b)$$

Values of y and z can be found by substituting x^2 in (7) and (8).