Three Equations

Submission deadline: April 29th 2020

Solve the following system of equations for $x, y$ and $z$

\[
\begin{align*}
  x^2 y^2 + x^2 z^2 &= axyz \\
  y^2 z^2 + y^2 x^2 &= bxyz \\
  z^2 x^2 + z^2 y^2 &= cxyz
\end{align*}
\]

where, $a, b$ and $c$ are given constants.

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Discussion:

\[ x^2y^2 + x^2z^2 = axyz \quad (1) \]
\[ y^2z^2 + y^2x^2 = bxyz \quad (2) \]
\[ z^2x^2 + z^2y^2 = cxyz \quad (3) \]

It is easy to see that if two of the variables are equal to 0, and the remaining one takes any value, then \( x, y, z \) is a solution. Moreover, it is not possible to have a solution where one variable is zero and the other two are non-zero. Therefore we will find solutions assuming each variable is non-zero.

Since \( x \neq 0 \), from (1) we get that

\[ x = a \frac{yz}{y^2 + z^2} \quad (4) \]

Substitute (4) in (2). Then, since \( yz \neq 0 \), we get

\[ a^2z^2 + (y^2 + z^2)^2 - ca(y^2 + z^2) = 0 \quad (5) \]

Next Substitute (4) in (3). Then, we get

\[ y^2 + z^2 = \frac{1}{2}a(b + c - a) \quad (6) \]

Similarly we can get

\[ z^2 + x^2 = \frac{1}{2}b(a + c - b) \quad (7) \]
\[ x^2 + y^2 = \frac{1}{2}c(a + b - c) \quad (8) \]

From (6) and (7) we get that \( x^2 - y^2 = (a - b)(a + b - c)/2 \). Thus,

\[ x^2 = \frac{1}{4}(a + b - c)(a + c - b) \]

Values of \( y \) and \( z \) can be found by substituting \( x^2 \) in (7) and (8).