Sines and more Sines.
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Evaluate

\[ \int_{0}^{\pi} \left( \frac{\sin(nx)}{\sin(x)} \right)^2 \, dx \]

where \( n \) is a positive integer.

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Discussion

Clearly, for $n > 1$,

$$\frac{\sin(nx)}{\sin(x)} = \frac{\sin((n-1)x) \cos(x) + \cos((n-1)x) \sin(x)}{\sin(x)}$$

thus

$$\frac{\sin(nx)}{\sin(x)} = \cos((n-1)x) + \frac{\cos(x)(\sin((n-2)x) \cos(x) + \cos((n-2)x) \sin(x))}{\sin(x)}$$

$$= \cos((n-1)x) + \cos((n-2)x) \cos(x) + \frac{\sin((n-2)x) \cos^2(x)}{\sin(x)}$$

$$= \cos((n-1)x) + \cos((n-2)x) \cos(x) - \sin((n-2)x) \sin(x) + \frac{\sin((n-2)x)}{\sin(x)}.$$  

We finally have that

$$\frac{\sin(nx)}{\sin(x)} = 2 \cos((n-1)x) + \frac{\sin((n-2)x)}{\sin(x)} \quad (1)$$

Let $I_n = \int_0^{\pi} \left(\frac{\sin(nx)}{\sin(x)}\right)^2 dx$. Then

$$I_n = 4 \int_0^{\pi} \cos^2((n-1)x) dx + 4 \int_0^{\pi} \frac{\cos((n-1)x) \sin((n-2)x)}{\sin(x)} dx + \int_0^{\pi} \left(\frac{\sin((n-2)x)}{\sin(x)}\right)^2 dx$$

Since $2 \cos((n-1)x) \sin((n-2)x) = \sin((2n-3)x) + \sin(-x)$ we get

$$I_n - I_{n-2} = 4 \int_0^{\pi} \cos^2((n-1)x) dx - 2 \int_0^{\pi} dx + 2 \int_0^{\pi} \frac{\sin((2n-3)x)}{\sin(x)} dx$$

Thus

$$I_n - I_{n-2} = 2 \int_0^{\pi} \frac{\sin((2n-3)x)}{\sin(x)} dx. \quad (2)$$

Next we will compute $J_n = \int_0^{\pi} \frac{\sin(nx)}{\sin(x)} dx$. Since

$$J_n = \int_0^{\pi} \frac{2 \cos((n-1)x) + \sin((n-2)x)}{\sin(x)} dx$$

Thus

$$J_n - J_{n-2} = \int_0^{\pi} 2 \cos((n-1)x) dx$$

Hence $J_n = J_{n-2}$ for $n > 2$. Clearly $J_1 = \pi$ and $J_2 = 0$, we get that $J_{2k} = 0$ and $J_{2k+1} = \pi$ for all positive integers $k$.

Therefore

$$I_n - I_{n-2} = 2\pi$$

And $I_1 = 1$ and $I_2 = 2\pi$, thus

$I_{2k} = 2k\pi$ and $I_{2k+1} = (2k + 1)\pi$.

About half of the solutions were variations of the solution above and the others solved the problem by expressing sine function using the exponential function.