Factors and Factorials

Submission deadline: August 28^{th} 2021

Describe all positive integers n such that $(n-1)!$ is not divisible by n.

The problem was solved by

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Discussion;

If n is a prime, it is clear that n cannot divide $(n-1)!$ since n does not appear in $1 \times 2 \times 3 \times \cdots \times (n-1)$ as a factor.

Now assume that n is a not a prime. Then, $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$ where each p_i is a prime.

If $k > 1$, then it is clear that $p_i^{n_i} < n$ for each i. Needless to say, $p_1, \cdots, p_k$ are distinct. Thus, each $p_i^{n_i}$ appears as a factor in $1 \times 2 \times 3 \times \cdots \times (n-1)$. Thus, n divides $(n-1)!$

If $k = 1$, then $n = p_1^{n_1}$. We only need to look at $n_1 > 1$. Clearly $n = p_1 \cdot p_1^{n_1 - 1}$. If $n_1 > 2$, then p_1 and $p_1^{n_1-1}$ are distinct. Since $p_1 < n$ and $p_1^{n_1-1} < n$, they both appear in $1 \times 2 \times 3 \times \cdots \times (n-1)$, hence n divides $(n-1)!$

Finally, we only need to look at $n = p_1^2$. If $2p_1 \leq n-1$, then both $2p_1$ and p_1 appear in $1 \times 2 \times 3 \times \cdots \times (n-1)$, hence p_1^2 is a factor of $(n-1)!$ Thus, n divides $(n-1)!$ If $2p_1 > n-1$, then $1 > p_1(p_1 - 2)$, and it easily follows that $p_1 = 2$.

Thus, $(n-1)!$ is not divisible by n if and only if, $n = 4$ or n is a prime.