Sum and sum of squares

Submission deadline: December $31^{\textbf{st}}$ 2017

Find infinitely many positive numbers $x_1, x_2, x_3, \cdots$ so that

$$x_1 + x_2 + x_3 + \cdots = 2017$$

and

$$x_1^2 + x_2^2 + x_3^2 + \cdots = 2017.$$

The problem was solved by

- Yousuf Abo Rahama, *American University of Sharjah, UAE.*
- Kamel Samara, *College of Medicine, Univeristy of Sharjah, UAE.*
- Daniel Horvath, *EduBase L.L.C, Hungary.*

Discussion;

Solution 1

The simplest way is to use a geometric series $x_n = ar^n$. If $-1 < r < 1$, then

$$x_1 + x_2 + x_3 + \cdots = \frac{ar}{1-r}$$

and

$$x_1^2 + x_2^2 + x_3^2 + \cdots = \frac{a^2r^2}{1-r^2}.$$

Now, by solving $ar/(1-r) = 2017$ and $a^2r^2/(1-r^2) = 2017$ for a, r we get

$$x_n = \frac{2017}{1008} \cdot \left(\frac{2016}{2018}\right)^n$$

Solution 2

Daniel Horvath wrote a very interesting alternative solution using the Riemann zeta function

$$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots$$

It is known that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. Now construct the series as following;

$$1 + 1 + \cdots + 1 + x_{2015} + x_{2016} + \frac{6}{\pi^2} + \frac{6}{\pi^2}\frac{1}{2^2} + \frac{6}{\pi^2}\frac{1}{3^2} + \cdots$$

where

$$x_{2015} = 1 - \sqrt{3/10} \quad \text{and } x_{2016} = 1 + \sqrt{3/10}.$$

Notice that each one of the first 2014 terms is 1. From the 2017^{th} term onwards, the terms of $\zeta(2)\dfrac{6}{\pi^2}$ are used.

It is not difficult to see that the second method would work with terms of many positive valued convergent series and not just the terms of the Riemann zeta function.