Fours, Eights and Nine.

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Show that each number of the sequence

$$49, 4489, 444889, 44448889, \cdots$$

is a perfect square.

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Discussion

The n^{th} term of this sequence has the digit 4 in the first n places followed by 8 in the next $n-1$ places and 9 at the end.

Let $q = 66\cdots 6$, be the integer with $n+1$ digits and each equal to 6. Thus $q = 6(10^n + 10^{n-1} + \cdots + 1)$. It is easy to see that

$$q = \frac{2}{3}(10^{n+1} - 1).$$

Therefore $q^2 = 4(10^{2(n+1)} - 2 \cdot 10^{n+1} + 1)/9$. Now,

$$q^2 = \frac{4}{9}((10^{2(n+1)} - 1) - 2(10^{n+1} - 1))$$

Upon factoring further, we get

$$q^2 = 4((10^{2n+1} + 10^{2n} + \cdots + 1) - 2(10^n + 10^{n-1} + \cdots + 1))$$

Now $q^2 + 2q = 4(10^{2n+1} + 10^{2n} + \cdots + 1) + 4(10^n + 10^{n-1} + \cdots + 1)$. Thus, $q^2 + 2q$ is the integer with the digit 4 in the first n places and 8 in the remaining places. Now it is easy to see that $(q+1)^2$ is the $n+1^{th}$ term in the sequence.