A Double Summation

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Evaluate

\[
\sum_{m=1}^{\infty} \sum_{n=1 \atop n \neq m}^{\infty} \frac{1}{m^2 - n^2}
\]

The problem was solved by

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Discussion:

It is easy to see that

\[
\sum_{m=1}^{\infty} \sum_{n=1, n \neq m}^{\infty} \frac{1}{m^2 - n^2} = \sum_{m=1}^{\infty} \frac{1}{2m} \sum_{n=1, n \neq m}^{\infty} \frac{1}{m + n} = \frac{1}{n - m}
\]

Let \(a_n = 1/(m + n)\) and \(b_n = 1/(n - m)\). Then it is clear that \(a_n = b_{2m+n}\). For \(n < 2m\), it is easy to see that \(b_n = -b_{2m-n}\). Thus,

\[
\sum_{n=1}^{\infty} a_n - b_n = -b_{2m} - b_{3m}
\]

Therefore

\[
\sum_{n=1}^{\infty} a_n - b_n = \frac{-3}{2m}
\]

Thus,

\[
\sum_{m=1}^{\infty} \sum_{n=1, n \neq m}^{\infty} \frac{1}{m^2 - n^2} = \sum_{m=1}^{\infty} \frac{1}{2m} \sum_{m=1}^{\infty} \frac{-3}{2m} = \frac{-3}{4} \sum_{m=1}^{\infty} \frac{1}{m^2}
\]

Which is equal to \(-\frac{\pi^2}{8}\)