A Double Summation

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Evaluate

$$\sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{1}{m^2 - n^2}$$

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Discussion;

It is easy to see that

$$\sum_{m=1}^{\infty}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{1}{m^2-n^2}=\sum_{m=1}^{\infty}\frac{1}{2m}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{1}{m+n}-\frac{1}{n-m}$$

Let $a_n = 1/(m+n)$ and $b_n = 1/(n-m)$. Then it is clear that $a_n = b_{2m+n}$. For $n < 2m$, it is easy to see that $b_n = -b_{2m-n}$. Thus,

$$\sum_{\substack{n=1\\n\neq m}}^{\infty} a_n - b_n = -b_{2m} - b_{3m}$$

Therefore

$$\sum_{\substack{n=1\\n\neq m}}^{\infty} a_n - b_n = \frac{-3}{2m}$$

Thus,

$$\sum_{m=1}^{\infty}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{1}{m^2-n^2}=\sum_{m=1}^{\infty}\frac{1}{2m}\frac{-3}{2m}=\frac{-3}{4}\sum_{m=1}^{\infty}\frac{1}{m^2}$$

Which is equal to $-\frac{\pi^2}{8}$