Subtract and Divide

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Find all positive integers a, b and c such that $1 < a < b < c$ and

$$(a-1)(b-1)(c-1) \text{ is a divisor of } abc - 1.$$

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Discussion:

It is easy to see that

$$(a-1)(b-1)(c-1) = (abc-1) - [a(b-1) + c(a-1) + b(c-1)]$$

thus,

$$1 = \frac{(abc-1)}{(a-1)(b-1)(c-1)} - \frac{[a(b-1)+c(a-1)+b(c-1)]}{(a-1)(b-1)(c-1)} \tag{1}$$

Therefore, it is clear that $(a-1)(b-1)(c-1)$ is a divisor of $abc-1$ if and only if $(a-1)(b-1)(c-1)$ is a divisor of $a(b-1)+c(a-1)+b(c-1)$. Clearly,

$$\frac{[a(b-1)+c(a-1)+b(c-1)]}{(a-1)(b-1)(c-1)} = \frac{a}{(a-1)(c-1)} + \frac{c}{(b-1)(c-1)} + \frac{b}{(a-1)(b-1)}$$

Since $a \geq 2, b \geq 3, c \geq 4$, it follows that $\dfrac{a}{(a-1)(c-1)} \leq \dfrac{2}{3}$, $\dfrac{c}{(c-1)(b-1)} \leq \dfrac{2}{3}$, and $\dfrac{b}{(b-1)(a-1)} \leq \dfrac{3}{2}$. Hence,

$$\frac{a}{(a-1)(c-1)} + \frac{c}{(b-1)(c-1)} + \frac{b}{(a-1)(b-1)} < 3$$

Let

$$m = \frac{a}{(a-1)(c-1)} + \frac{c}{(b-1)(c-1)} + \frac{b}{(a-1)(b-1)} \tag{2}$$

Thus, if m is an integer it can only take the values 1 or 2. Next we will consider the two cases.

Case 1: $m = 2$.

First assume that $a \geq 3$. Then, $b \geq 4$ and $c \geq 5$. Hence,

$a/((a-1)(c-1)) \leq 3/8$, $c/((c-1)(b-1)) \leq 5/12$, and $b/((b-1)(a-1)) \leq 2/3$

Addition of the fractions above results in a value less than 2. Thus $a \geq 3$, is false. Since $a > 1$, it is clear that the only possible value for a is 2.

Now assume that $b \geq 5$. Then $c \geq 6$. Now we have,

$a/((a-1)(c-1)) \leq 2/5$, $c/((c-1)(b-1)) \leq 6/20$ and $b/((b-1)(a-1)) \leq 5/4$

The sum of the fractions above again results in a value less than 2, thus it shows that $b < 5$. Since $a = 2$, the only possible values for b are 3 or 4.

If $b = 4$, from (2) it follows that $c = 8$.

From (2) it can be seen that $b = 3$ is not possible.

Thus, when $m = 2$, the only solution is $a = 2, b = 4$ and $c = 8$.

Case 2: $m = 1$.

If $m = 1$, then from (1) it follows that $abc - 1 = 2(a-1)(b-1)(c-1)$. Thus, abc must be odd, therefore, each of a, b and c must be an odd integer.

If $a \geq 5$, then $b \geq 7$ and $c \geq 9$. Now we have that

$a/((a-1)(c-1)) \leq 5/32$, $c/((c-1)(b-1)) \leq 3/16$ and $b/((b-1)(c-1)) \leq 7/48$. Sum of the fractions above is less than 1.

Therefore $a < 5$. Hence the only possible value for a is 3.

Now if $b \geq 7$, then $c \geq 9$, and hence we have

$b/((b-1)(a-1)) \leq 7/12$, $c/((c-1)(b-1)) \leq 3/16$ and $a/((a-1)(c-1)) \leq 3/16$ and the fractions above add up to a value less than 1. Hence it is clear that the only possible value for b is 5. Using $a = 3, b = 5$ in $abc-1 = 2(a-1)(b-1)(c-1)$ we get that $c = 15$.

Thus, when $m = 1$, the only solution is $a = 3, b = 5$ and $c = 15$.