Subtract and Divide
Submission deadline: March 29th 2021

Find all positive integers $a, b$ and $c$ such that $1 < a < b < c$ and

$$(a - 1)(b - 1)(c - 1) \text{ is a divisor of } abc - 1.$$ 

The problem was solved by

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Discussion:
It is easy to see that

\[(a - 1)(b - 1)(c - 1) = (abc - 1) - [a(b - 1) + c(a - 1) + b(c - 1)]\]

thus,

\[1 = \frac{(abc - 1)}{(a - 1)(b - 1)(c - 1)} - \frac{[a(b - 1) + c(a - 1) + b(c - 1)]}{(a - 1)(b - 1)(c - 1)}\]

(1)

Therefore, it is clear that \((a - 1)(b - 1)(c - 1)\) is a divisor of \(abc - 1\) if and only if \((a - 1)(b - 1)(c - 1)\) is a divisor of \(a(b - 1) + c(a - 1) + b(c - 1)\). Clearly,

\[
\frac{[a(b - 1) + c(a - 1) + b(c - 1)]}{(a - 1)(b - 1)(c - 1)} = \frac{a}{(a - 1)(c - 1)} + \frac{c}{(b - 1)(c - 1)} + \frac{b}{(a - 1)(b - 1)}
\]

Since \(a \geq 2, b \geq 3, c \geq 4\), it follows that

\[
\frac{a}{(a - 1)(c - 1)} \leq \frac{2}{3}, \quad \frac{c}{(c - 1)(b - 1)} \leq \frac{2}{3}, \quad \text{and} \quad \frac{b}{(b - 1)(a - 1)} \leq \frac{3}{2}
\]

Hence,

\[
\frac{a}{(a - 1)(c - 1)} + \frac{c}{(b - 1)(c - 1)} + \frac{b}{(a - 1)(b - 1)} < 3
\]

Thus, if \(m\) is an integer it can only take the values 1 or 2. Next we will consider the two cases.

**Case 1: \(m = 2\).**
First assume that \(a \geq 3\). Then, \(b \geq 4\) and \(c \geq 5\). Hence,

\[
a/((a-1)(c-1)) \leq 3/8, \quad c/((c-1)(b-1)) \leq 5/12, \quad \text{and} \quad b/((b-1)(a-1)) \leq 2/3
\]

Addition of the fractions above results in a value less than 2. Thus \(a \geq 3\) is false. Since \(a > 1\), it is clear that the only possible value for \(a\) is 2.

Now assume that \(b \geq 5\). Then \(c \geq 6\). Now we have,

\[
a/((a-1)(c-1)) \leq 2/5, \quad c/((c-1)(b-1)) \leq 6/20 \quad \text{and} \quad b/((b-1)(a-1)) \leq 5/4
\]

The sum of the fractions above again results in a value less than 2, thus it shows that \(b < 5\). Since \(a = 2\), the only possible values for \(b\) are 3 or 4.

If \(b = 4\), from (2) it follows that \(c = 8\).

From (2) it can be seen that \(b = 3\) is not possible.

**Thus, when \(m = 2\), the only solution is \(a = 2, b = 4\) and \(c = 8\).**

**Case 2: \(m = 1\).**
If \(m = 1\), then from (1) it follows that \(abc - 1 = 2(a - 1)(b - 1)(c - 1)\). Thus, \(abc\) must be odd, therefore, each of \(a, b\) and \(c\) must be an odd integer.

If \(a \geq 5\), then \(b \geq 7\) and \(c \geq 9\). Now we have that

\[
a/((a-1)(c-1)) \leq 5/32, \quad c/((c-1)(b-1)) \leq 3/16 \quad \text{and} \quad b/((b-1)(c-1)) \leq 7/48
\]

Sum of the fractions above is less than 1.
Therefore $a < 5$. Hence the only possible value for $a$ is 3.

Now if $b \geq 7$, then $c \geq 9$, and hence we have

\[ \frac{b}{((b-1)(a-1))} \leq \frac{7}{12}, \quad \frac{c}{((c-1)(b-1))} \leq \frac{3}{16} \quad \text{and} \quad \frac{a}{((a-1)(c-1))} \leq \frac{3}{16} \]

and the fractions above add up to a value less than 1. Hence it is clear that the only possible value for $b$ is 5. Using $a = 3, b = 5$ in $abc − 1 = 2(a−1)(b−1)(c−1)$ we get that $c = 15$.

Thus, when $m = 1$, the only solution is $a = 3, b = 5$ and $c = 15$. 