Multiples of 5

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Describe all positive integers $n$ such that $n^5 - n$ is divisible by 5.

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Discussion:
First assume that \( n = 2k \) for some positive integer \( k \).
It is easy to see that \( n^5 - n = 2k \cdot (2k - 1) \cdot (2k + 1) \cdot (4k^2 + 1) \). Thus, \( n^5 - n = 2k \cdot (2k - 1) \cdot (2k + 1) \cdot (4k^2 - 4 + 5) \). Therefore, \( n^5 - n \) is equal to,
\[
2k \cdot (2k - 1) \cdot (2k + 1) \cdot (2k + 2) \cdot (2k - 2) + 5 \cdot 2k \cdot (2k - 1) \cdot (2k + 1)
\]
In the summation above, the first term is a product of five consecutive integers, hence is divisible by 5. The second term is clearly divisible by 5. Thus, when \( n \) is even, \( n^5 - n \) is divisible by 5.

Next assume that \( n = 2k + 1 \) for some positive integer \( k \). Clearly \( n^5 - n = (2k + 1) \cdot (2k) \cdot (2k + 2) \cdot (4k^2 + 4k + 2) \). Thus
\[
 n^5 - n = (2k + 1) \cdot (2k) \cdot (2k + 2) \cdot ((2k + 3)(2k - 1) + 5)
\]
Therefore,
\[
 n^5 - n = (2k + 1) \cdot (2k) \cdot (2k + 2) \cdot (2k + 3) \cdot (2k - 1) + 5(2k + 1) \cdot (2k) \cdot (2k + 2)
\]
The first term on the right hand side is a product of 5 consecutive integers, therefore is divisible by 5 and second term is clearly divisible by 5. Hence when \( n \) is odd, \( n^5 - n \) is divisible by 5.
Therefore \( n^5 - n \) is divisible by 5 for all integers \( n \).