Multiplicativity

Submission deadline: February 27^{th} 2023

Let f be a function defined on positive integers that takes integer values with the following properties.

1. $f(2) = 2$
2. $f(mn) = f(m)f(n)$ for all m and n
3. $f(m) > f(n)$ whenever $m > n$.

Find $f(n)$ for $n = 1, 2, \cdots$

The problem was solved by

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Discussion:

Since $f(2) = 2$, from the condition $f(mn) = f(m)f(n)$ it follows that $f(2^k) = 2^k$, for all natural numbers k. Now consider all the integers $r_1, r_2, \cdots, r_{2^k-1}$ between 2^k and 2^{k+1}, labelled such that $r_i = 2^k + i$. Then,

$$2^k < r_1 < r_2 < \cdots < r_{2^k-1} < 2^{k+1}$$

and hence,

$$2^k < f(r_1) < f(r_2) < \cdots < f(r_{2^k-1}) < 2^{k+1}$$

Since $f(r_i)$ is a natural number for each i, it easily follows from above that $f(r_i) = r_i$.

Letting $m = 2$ and $n = 1$ in the condition $f(mn) = f(m)f(n)$ we get that $f(1) = 1$. Thus $f(n) = n$ for all natural numbers.