Combinations

Submission deadline: June $29^{\textbf{th}}$ 2022

For $0 \leq k \leq n$, let ${}^nC_k = \dfrac{n!}{k! \cdot (n-k)!}$. Find

$$^nC_k +^{n+1} C_k + \cdots +^{n+m} C_k$$

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Discussion.

Note that

$$(1+x)^n = \sum_{j=0}^{n} {}^nC_j x^j$$

Let $f(x) = (1+x)^n + (1+x)^{n+1} + \cdots + (1+x)^{n+m}$. Then it is clear that ${}^nC_k +^{n+1} C_k + \cdots +^{n+m} C_k$ is the coefficient of x^k term of $f(x)$. Since f is a geometric series it is easy to see that

$$f(x) = \frac{1}{x}((1+x)^{n+m+1} - (1+x)^n)$$

If $k < n$, the coefficient of x^k, is ${}^{n+m+1}C_{k+1} -^n C_{k+1}$. If $k = n$, then the coefficient is ${}^{n+m+1}C_{k+1}$.