Polynomial Factoring

Submission deadline: September $29^{\textbf{th}}$ 2022

Prove that for $m = 0, 1, 2, \cdots$

$$S_m(k) = 1 + 2^{2m+1} + \cdots + k^{2m+1}$$

is a polynomial in $k(k+1)$.

We did not receive any correct solutions.

Discussion.

We use induction. If $m = 0$, then $S_0(k) = k(k+1)/2$. Assume that $S_m(k)$ is a polynomial in $k(k+1)$ up to $m-1$. We have that

$$(n+1)^{m+1} - (n-1)^{m+1} = \sum_{j=0}^{m+1} \binom{m+1}{j} n^{m+1-j} - \sum_{j=0}^{m+1} \binom{m+1}{j} n^{m+1-j}(-1)^j$$
$$= 2\sum_{p=0}^{s} \binom{m+1}{2p+1} n^{m+1-(2p+1)}$$

where $2s+1 = m+1$, if $m+1$ is odd and $2s+1 = m$, if $m+1$ is even. Thus,

$$n^{m+1}(n+1)^{m+1} - n^{m+1}(n-1)^{m+1} = 2\sum_{p=0}^{s} \binom{m+1}{2p+1} n^{2m+2-(2p+1)}$$

Therefore,

$$(n(n+1))^{m+1} - (n(n-1))^{m+1} = 2(m+1)n^{2m+1} + 2\sum_{p=1}^{s} \binom{m+1}{2p+1} n^{2(m-p)+1}$$

If $a_n = (n(n+1))^{m+1} - (n(n-1))^{m+1}$, then

$$\sum_{n=1}^{k} a_n = 2(m+1)\sum_{n=1}^{k} n^{2m+1} + 2\sum_{p=1}^{s} \binom{m+1}{2p+1} \sum_{n=1}^{k} n^{2(m-p)+1}$$

It is easy to see that $\sum_{n=1}^{k} a_n = (k(k+1))^{m+1}$. Hence we have

$$(k(k+1))^{m+1} = 2(m+1)S_m(k) + \sum_{p=1}^{s} \binom{m+1}{2p+1} S_{m-p}(k)$$

Since $S_{m-p}(k)$ is a polynomial for $1 \leq p \leq s$, the desired result follows from mathematical induction.