Rational or Irrational?

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Is $\sin(1^\circ)$ an irrational number or a rational number?

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Discussion:

Assume that $r = \sin(1^\circ)$, is rational. Since $\cos(2\theta) = 1 - 2\sin^2(\theta)$, we get $\cos(2^\circ) = 1 - 2r^2$. Moreover, $\cos(2\theta) = 2\cos^2(\theta) - 1$, hence it follows that $\cos(4^\circ) = 2(1-2r^2)^2 - 1$, and is rational. Thus, repeatedly doubling the angle 5 times we see that $s = \cos(32^\circ)$ is rational. Since $\cos(30^\circ + 2^\circ) = \cos(30^\circ)\cos(2^\circ) - \sin(30^\circ)\sin(2^\circ)$, and $\sin(2^\circ) = 2\sin(1^\circ)\cos(1^\circ)$, it follows that

$$s = \frac{\sqrt{3}}{2}(1-2r^2) - r\sqrt{1-r^2}.$$

Taking the first term of the right hand side of the equation above to the other side and squaring the equation yields

$$s^2 - \sqrt{3}s(1-2r^2) + \frac{3}{4}(1-2r^2)^2 = r^2(1-r^2)$$

Since s and r are rational numbers, it follows from the equation above that $\sqrt{3}$ is rational. Thus, $\sin(1^\circ)$ cannot be a rational number.