Choices

Submission deadline: November 28^{th} 2023

Denote $\dfrac{n!}{k!(n-k)!}$ by $\dbinom{n}{k}$ where $0 \le k \le n$.
Evaluate

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots$$

The series is continued up to the point where the lower number becomes greater than the upper number.

The problem was solved by

- Rohan Mitra, *Alumni, American University of Sharjah, UAE.*

- Hari Kishan, *Department of Mathematics D.N. College, Meerut, India.*

- Roland Karlsson, *Technical University of Linkoping, Sweden.*

Discussion:

Clearly, $(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k$

If k is odd, then $k = 4p+1$ or $k = 4p+3$, where p is a natural number. Thus, we have that $i^k = i$ or $i^k = -i$.

If k is even, then $k = 4p+2$ or $k = 4p$, where p is a natural number and it is easy to see that $i^{4p+2} = -1$ and $i^{4p} = 1$.

Thus, it follows that the required sum is the real part of $(1+i)^n$. Since $1+i = \sqrt{2}e^{i\pi/4}$, we get

$$(\sqrt{2})^n \cos\left(n\frac{\pi}{4}\right) = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots + (-1)^m \binom{n}{2m}$$

where $2m \leq n < 2(m+1)$.